

# Thermal entanglement and teleportation in a two-qubit Heisenberg chain with Dzyaloshinski-Moriya anisotropic antisymmetric interaction

Guo-Feng Zhang<sup>\*†</sup>

*Department of physics, School of sciences,  
Beijing University of Aeronautics and Astronautics,  
Xueyuan Road No. 37, Beijing 100083, People's Republic of China*

## Abstract

Thermal entanglement of a two-qubit Heisenberg chain in presence of the Dzyaloshinski-Moriya (DM) anisotropic antisymmetric interaction and entanglement teleportation when using two independent Heisenberg chains as quantum channel are investigated. It is found that the DM interaction can excite the entanglement and teleportation fidelity. The output entanglement increases linearly with increasing value of input one, its dependences on the temperature, DM interaction and spin coupling constant are given in detail. Entanglement teleportation will be better realized via antiferromagnetic spin chain when the DM interaction is turned off and the temperature is low. However, the introduction of DM interaction can cause the ferromagnetic spin chain to be a better quantum channel for teleportation. A minimal entanglement of the thermal state in the model is needed to realize the entanglement teleportation regardless of antiferromagnetic or ferromagnetic spin chains.

PACS numbers: 03.67.Hk, 03.65.Ud, 75.10.Jm

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<sup>\*</sup> Corresponding author.

<sup>†</sup> Email: gf1978zhang@buaa.edu.cn

## I. INTRODUCTION

Entanglement is one of the most fascinating features of quantum mechanics and plays a central role in quantum information processing. In recent years, there has been an ongoing effort to characterize qualitatively and quantitatively the entanglement properties of condensed matter systems and apply them in quantum information. The quantum entanglement in solid state systems such as spin chains is an important emerging field [1, 2, 3, 4, 5, 6, 7, 8]. Spin chains are natural candidates for the realization of the entanglement compared with the other physics systems. The Heisenberg chain, the simplest spin chain, has been used to construct a quantum computer[9]. By suitable coding, the Heisenberg interaction alone can be used for quantum computation [10, 11, 12]. In addition, quantum teleportation has been extensively investigated both experimentally and theoretically. Since the decoherence from environment always impacts on the degree of entanglement, the resource of maximally entangled states is hard to prepare in a real experiment. Certainly, a mixed entangled state as the resource is approximately near to the real circumstances. As an important source, the thermal entanglement has been widely investigated in many previous studies. Also the entanglement teleportation via thermal entangled states of a two-qubit Heisenberg XX chain has been reported [13]. Yeo [14] *et al* studied the influence of anisotropy and magnetic field on quantum teleportation via Heisenberg XY chain. But only the spin-spin interaction was considered in those studies, the effects of spin-orbit coupling on the entanglement and teleportation are rarely concerned. These are the motivations of this paper.

In this paper, we investigate the influence of spin-orbit coupling on the thermal entanglement. The information transmission by a pair of thermal mixed states in a two-qubit Heisenberg chain in presence of the DM anisotropic antisymmetric interaction is investigated. A minimal entanglement in the quantum channel is needed to transfer entanglement information. Thermal entanglement will be given in Sec. II. The entanglement teleportation of two-qubit pure states and its fidelity are derived in Sec. III and IV. In Sec.V a discussion concludes the paper.

## II. THE EFFECT OF DM INTERACTION ON THERMAL ENTANGLEMENT

In this paper, we consider the Heisenberg model with DM interaction, which can be described by

$$H_{DM} = \frac{J}{2}[(\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z}) + \vec{D} \cdot (\vec{\sigma}_1 \times \vec{\sigma}_2)], \quad (1)$$

here  $J$  is the real coupling coefficient and  $\vec{D}$  is the DM vector coupling. The DM anisotropic antisymmetric interaction arises from spin-orbit coupling [15, 16]. The coupling constant  $J > 0$  corresponds to the antiferromagnetic case and  $J < 0$  to the ferromagnetic case. For simplicity, we choose  $\vec{D} = D\vec{z}$ , then the Hamiltonian  $H_{DM}$  becomes

$$\begin{aligned} H_{DM} &= \frac{J}{2}[\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z} + D(\sigma_{1x}\sigma_{2y} - \sigma_{1y}\sigma_{2x})] \\ &= J[(1 + iD)\sigma_{1+}\sigma_{2-} + (1 - iD)\sigma_{1-}\sigma_{2+}]. \end{aligned} \quad (2)$$

Without loss of generality, we define  $|0\rangle$  ( $|1\rangle$ ) as the ground (excited) state of a two-level particle. The eigenvalues and eigenvectors of  $H_{DM}$  are given by

$$\begin{aligned} H_{DM}|00\rangle &= \frac{J}{2}|00\rangle, \\ H_{DM}|11\rangle &= \frac{J}{2}|11\rangle, \\ H_{DM}|+\rangle &= (J\sqrt{1+D^2} - \frac{J}{2})|+\rangle, \\ H_{DM}|-\rangle &= (-J\sqrt{1+D^2} - \frac{J}{2})|-\rangle, \end{aligned} \quad (3)$$

where  $|\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm e^{i\theta}|10\rangle)$  and  $\theta = \arctan D$ .

As the thermal fluctuation is introduced into the system, the state of a typical solid-state system at thermal equilibrium (temperature  $T$ ) is  $\rho(T) = \frac{1}{Z}e^{-\beta H}$ , where  $H$  is the Hamiltonian,  $Z = \text{tr}e^{-\beta H}$  is the partition function, in the standard basis  $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$ , the density matrix  $\rho(T)$  can be expressed as

$$\rho(T) = \frac{1}{Z} \begin{pmatrix} e^{-\frac{\beta J}{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{2}e^{\frac{1}{2}\beta(J-\delta)}(1 + e^{\beta\delta}) & \frac{1}{2}e^{i\theta}e^{\frac{1}{2}\beta(J-\delta)}(1 - e^{\beta\delta}) & 0 \\ 0 & \frac{1}{2}e^{-i\theta}e^{\frac{1}{2}\beta(J-\delta)}(1 - e^{\beta\delta}) & \frac{1}{2}e^{\frac{1}{2}\beta(J-\delta)}(1 + e^{\beta\delta}) & 0 \\ 0 & 0 & 0 & e^{-\frac{\beta J}{2}} \end{pmatrix}, \quad (4)$$

where  $Z = 2e^{-\frac{\beta J}{2}}(1 + e^{\beta J} \cosh \frac{\beta\delta}{2})$ ,  $\beta = \frac{1}{kT}$  and  $\delta = 2J\sqrt{1+D^2}$ . In the following calculation, we will write the Boltzmann constant  $k = 1$ . The entanglement of two qubits can be measured

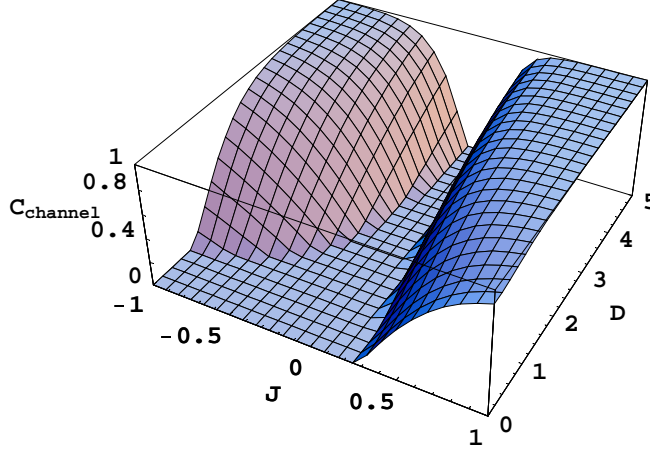


FIG. 1: (Color online) The thermal concurrence for the spin channel when  $T = 0.5$ .  $T$  is plotted in units of the Boltzmanns constant  $k$ . We work in units where  $D$  and  $J$  are dimensionless.

by the concurrence  $C$  which is defined as  $C = \max[0, 2 \max[\lambda_i] - \sum_i \lambda_i]$  [17], where  $\lambda_i$  are the square roots of the eigenvalues of the matrix  $R = \rho S \rho^* S$ ,  $\rho$  is the density matrix,  $S = \sigma_{1y} \otimes \sigma_{2y}$  and  $*$  stands for the complex conjugate. The concurrence is available, no matter whether  $\rho$  is pure or mixed. Note that we are working in units so that  $D$  and  $J$  are dimensionless.

Based on the definition of concurrence, we can obtain the concurrence at the finite temperature

$$C_{channel} \equiv C[\rho(T)] = \frac{2}{Z} \max \left\{ \frac{1}{2} |e^{\frac{1}{2}\beta(J-\delta)}(1 - e^{\beta\delta})| - e^{-\frac{\beta J}{2}}, 0 \right\}. \quad (5)$$

The concurrence  $C = 0$  indicates the vanishing entanglement. The critical temperature  $T_c$  above which the concurrence is zero is determined by the nonlinear equation

$$\begin{cases} e^{\frac{J}{T}} \sinh \frac{\delta}{2T} = -1, & \text{if } J < 0; \\ e^{\frac{J}{T}} \sinh \frac{\delta}{2T} = 1, & \text{if } J > 0. \end{cases} \quad (6)$$

which can be solved numerically. When  $D = 0$ , i.e.  $\delta = 2J$ , it is found that  $T_c = \frac{2J}{\ln 3}$  for  $J > 0$ , but there is no entanglement at any temperature for  $J < 0$ . These accord to the conclusions in Ref. [18]. Fig.1 demonstrates the dependence of thermal entanglement on  $J$  and  $D$  at  $T = 0.5$ . Although there is no entanglement for ferromagnetic case when  $D = 0$ , while  $D$  increases, entanglement will be inspired and the area of  $J$  for which  $C = 0$  will decrease. The entanglement can reach the maximum value by adjusting the DM interaction constant for the two cases.

### III. THERMAL ENTANGLEMENT TELEPORTATION

For the entanglement teleportation of the whole two-qubit system as the resource of the thermal mixed state in a Heisenberg spin chain, the standard teleportation through mixed states can be regarded as a general depolarising channel [19, 20]. Similar to the standard teleportation, the entanglement teleportation for the mixed channel of an input entangled state is destroyed and its replica state appears at the remote place after applying local measurement in the form of linear operators. We consider as input a qubit in an arbitrary pure state  $|\psi\rangle_{in} = \cos\frac{\theta}{2}|10\rangle + e^{i\phi}\sin\frac{\theta}{2}|01\rangle$  ( $0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$ ). Here different values of  $\theta$  describe all states with different amplitudes, and  $\phi$  stands for the phase of these states. The output state is then given by [21]

$$\rho_{out} = \sum_{ij} p_{ij}(\sigma_i \otimes \sigma_j) \rho_{in}(\sigma_i \otimes \sigma_j), \quad (7)$$

where  $\sigma_i$  ( $i = 0, x, y, z$ ) signify unit matrix  $I$  and three components of Pauli matrix  $\vec{\sigma}$  correspondingly,  $p_{ij} = \text{tr}(E^i \rho(T)) \cdot \text{tr}(E^j \rho(T))$  and  $\sum_{ij} p_{ij} = 1$ ,  $\rho_{in} = |\psi\rangle_{in} \langle \psi|$ . Here  $E^0 = |\Psi^-\rangle \langle \Psi^-|$ ,  $E^1 = |\Phi^-\rangle \langle \Phi^-|$ ,  $E^2 = |\Phi^+\rangle \langle \Phi^+|$ ,  $E^3 = |\Psi^+\rangle \langle \Psi^+|$ , in which  $|\Psi^\pm\rangle = (1/\sqrt{2})(|01\rangle \pm |10\rangle)$ ,  $|\Phi^\pm\rangle = (1/\sqrt{2})(|00\rangle \pm |11\rangle)$ .

It follows that the concurrence of initial state  $|\psi\rangle_{in}$  is  $C_{in} = 2|\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{i\phi}|$ . We calculate the measure of entanglement for the teleported state  $\rho_{out}$  to be

$$C[\rho_{out}] \equiv C_{out} = \max \left\{ \frac{2 \{ C_{in} e^{\beta J} (\sinh \frac{\beta \delta}{2})^2 - 2(1 + D^2) \cosh \frac{\beta \delta}{2} \}}{Z^2(1 + D^2)}, 0 \right\}. \quad (8)$$

From Eq.(8), it can be seen that  $C_{out}$  increases linearly with the increasing value of  $C_{in}$ . The result can also be seen from Fig. 2(a) and Fig.3.

The quantity  $C_{out}$  as a function of  $C_{in}$  is plotted in Fig. 2 when the DM interaction  $D$ , the temperature  $T$  and the coupling coefficient  $J$  are changed. Fig. 2(a) is a plot of  $C_{out}$  as functions of  $C_{in}$ ,  $T$  when  $D = 0$  and  $J = 1$  for which the critical temperature of the channel concurrence  $T_c = \frac{2J}{\ln 3} = \frac{2}{\ln 3} \approx 1.82$ , from Fig. 2(a), we know that  $C_{out}$  remains zero when  $T > 1$ , so a minimal entanglement of the thermal mixed state must be provided in such quantum channel in order to realize entanglement teleportation. When the initial state is in a maximum entangled state which corresponds to Fig.2(b),  $C_{out}$  exists regardless of the sign of  $J$ . For  $J < 0$ , firstly the output entanglement increases with the increasing  $D$  from zero to a certain value that is much smaller than  $C_{in}$  and then begins

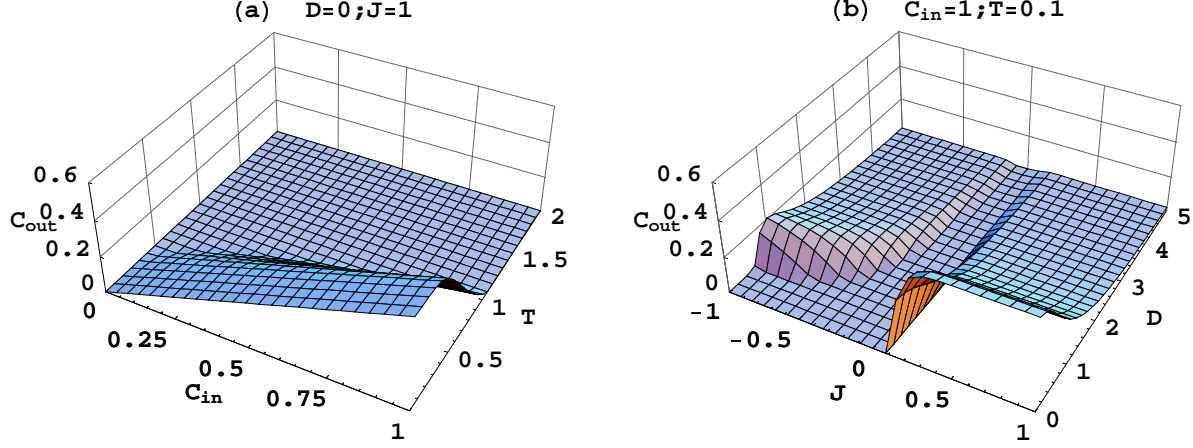


FIG. 2: (Color online) The teleported thermal concurrence  $C_{out}$  as a function of the input concurrence  $C_{in}$ , DM interaction  $D$ , spin coupling  $J$  and temperature  $T$ .  $T$  is plotted in units of the Boltzmanns constant  $k$ . We work in units where  $D$  and  $J$  are dimensionless.

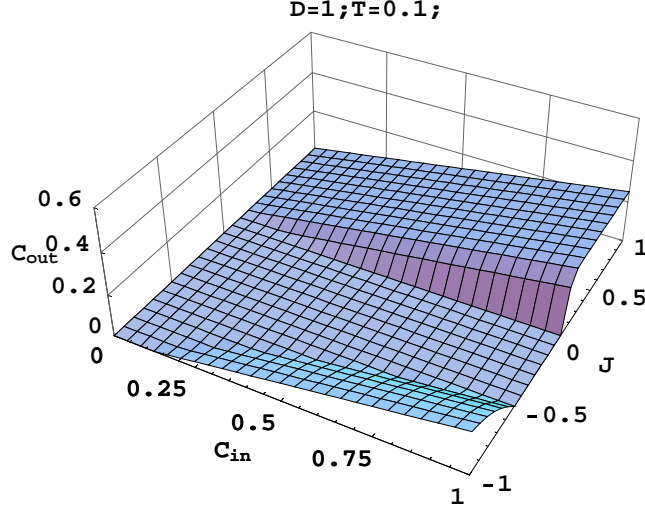


FIG. 3: (Color online) The teleported thermal concurrence  $C_{out}$  as a function of the input concurrence  $C_{in}$  and spin coupling  $J$  when temperature  $T = 0.1$  and DM interaction  $D = 1$ .  $T$  is plotted in units of the Boltzmanns constant  $k$ . We work in units where  $D$  and  $J$  are dimensionless.

to fall until to be zero. However,  $C_{out}$  decreases monotonously with the increasing  $D$  for  $J > 0$ . It may be advantageous for increasing  $C_{out}$  and the channel entanglement  $C_{channel}$  by introducing the DM interaction for  $J < 0$ , however, when  $J > 0$  the DM interaction can only cause  $C_{channel}$  increase. As the increase of DM interaction,  $C_{out}$  will decrease until

to be zero when  $D$  is large for both  $J > 0$  and  $J < 0$ . These are due to the fact that  $C_{out} = \max\{-4 \cosh[\beta\delta/2]/Z^2, 0\} = 0$  when  $D \rightarrow \infty$ . The maximum value of  $C_{out}$  is much smaller than that of the channel entanglement. Under the general circumstances, the output entanglement of two-qubit state  $|\psi\rangle_{in}$  will decrease via the quantum channel. These results can be found by comparing Fig.2 with Fig.1 [22].

Fig.3 shows the dependence of  $C_{out}$  on  $C_{in}$  and spin coupling  $J$  for a given DM interaction and temperature. As the channel concurrence shows,  $C_{out}$  behaves obviously different for  $J > 0$  and  $J < 0$ . For  $J < 0$ , only when  $|J| > 0.5$ ,  $C_{out}$  is nonvanishing when the initial state is a maximum entangled state. If  $0 < C_{in} < 1$ ,  $|J|$  must be larger in order to realize entanglement teleportation. We can know that  $C_{channel} \approx 0.597$  for  $J = -0.5$  at the same condition with Fig.3, these results show again a minimum entanglement must be provided. The same conclusion can be obtained for  $J > 0$ .

#### IV. THE FIDELITY OF ENTANGLEMENT TELEPORTATION

To characterize the quality of the teleported state  $\rho_{out}$ , it is often quite useful to look at the fidelity between  $\rho_{out}$  and  $\rho_{in}$  defined by [23]

$$F(\rho_{in}, \rho_{out}) = \left\{ \text{tr} \left[ \sqrt{(\rho_{in})^{1/2} \rho_{out} (\rho_{in})^{1/2}} \right] \right\}^2. \quad (9)$$

The concept of fidelity has been a useful indicator of the teleportation performance of a quantum channel when the input state is a pure state. The average fidelity  $F_A$  of teleportation can be formulated by

$$F_A = \frac{\int_0^{2\pi} d\phi \int_0^\pi F \sin \theta d\theta}{4\pi}. \quad (10)$$

If our model is used as quantum channel,  $F_A$  can be expressed by

$$F_A = \frac{2(1 + D^2) + e^{2\beta J}[1 + 2D^2 + (3 + 2D^2) \cosh \beta\delta]}{6(1 + D^2)(1 + e^{\beta J} \cosh \frac{\beta\delta}{2})^2}. \quad (11)$$

This is the maximal fidelity achievable from  $\rho(T)$ . In order to transmit  $|\psi\rangle_{in}$  with better fidelity than any classical communication protocol, we require Eq.(11) to be strictly greater than  $2/3 (\approx 0.667)$ . When  $D = 0$ , this requirement becomes  $e^{2\beta J} > 11$ .

The average fidelity  $F_A$  is plotted as a function of spin coupling  $J$  and temperature  $T$  when  $D = 0$  in Fig.4. When  $D = 0$ ,  $F_A$  is larger than  $2/3$  if  $0 < T < \frac{2J}{\ln 11}$ . So  $F_A$  is always

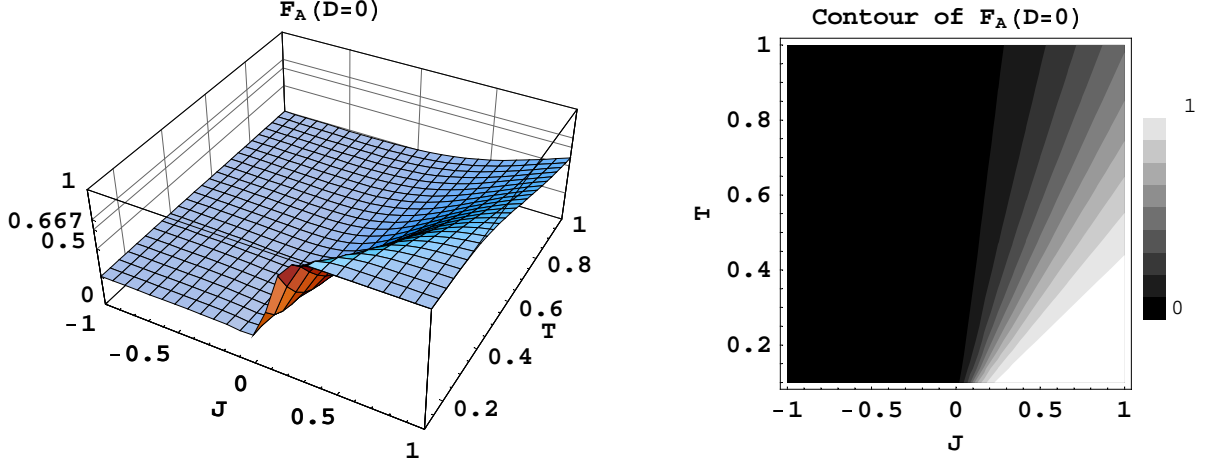


FIG. 4: (Color online) The average fidelity  $F_A$  as a function of spin coupling  $J$  and temperature  $T$  when  $D = 0$ .  $T$  is plotted in units of the Boltzmanns constant  $k$ . We work in units where  $D$  and  $J$  are dimensionless.

smaller than  $2/3$  for  $J < 0$  at any temperature. However, for  $J > 0$ ,  $F_A$  can arrive 1 at near zero temperature and begins to fall until  $2/3$  at the point  $T = \frac{2J}{\ln 11}$ . It means that the entanglement teleportation of the mixed channel is inferior to the classical communication when  $J < 0$  without DM interaction. Fig.5 give the dependence of  $F_A$  on DM interaction  $D$  and spin coupling  $J$ . By introducing the DM interaction,  $F_A$  can be larger than  $2/3$  for  $J < 0$  (for example,  $T = 0.1$ ,  $J \in [-0.5, -1]$ ). For  $J > 0$ ,  $F_A$  decreases monotonously with the increasing  $D$ . When the DM interaction is very strong,  $F_A$  approaches infinitely the value of  $2/3$  for both the two cases. These results show that we must strengthen the DM interaction to be a certain value in order to use ferromagnetic spin chain as a quantum channel for entanglement teleportation, which is contrary to antiferromagnetic spin chain.

## V. CONCLUSIONS

We have investigated the thermal entanglement of two-qubit spin chain with DM anisotropic antisymmetric interaction and entanglement teleportation via the model. The entanglement can reach the maximum value by adjusting the DM interaction constant for ferromagnetic and antiferromagnetic case. By introducing the DM interaction, the output entanglement and fidelity can be increased for ferromagnetic case, which are contrary to



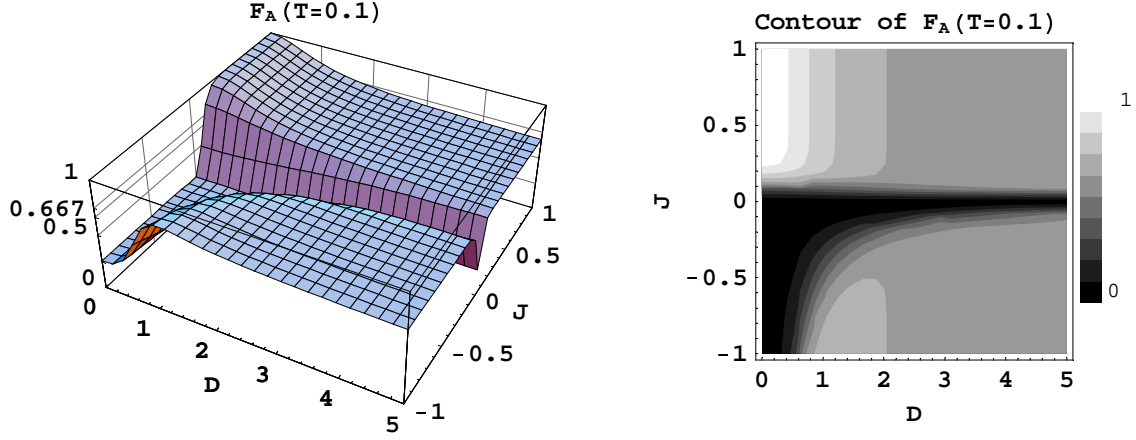


FIG. 5: (Color online) The average fidelity  $F_A$  as a function of  $J$  and  $D$  for a given temperature.  $T$  is plotted in units of the Boltzmanns constant  $k$ . We work in units where  $D$  and  $J$  are dimensionless.

antiferromagnetic case. When the DM interaction is very strong, the average fidelity of entanglement teleportation will approach a fixed value that is the maximal one for classical communication. A minimal entanglement of the thermal state in the model is needed to realize the entanglement teleportation.

## VI. ACKNOWLEDGEMENTS

This work was supported by the National Natural Science Foundation of China (Grant No. 10604053).

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